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Layered Convective Flows of Vertically Swirling Incompressible Fluid Affected by Tangential Stresses

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Abstract. Using the construction of several particular exact solutions, the article investigates the overdetermined system of equations of thermal convection of a vertically swirling viscous fluid in the Boussinesq approximation. The vertical component of vorticity in a non-rotating fluid is generated by an inhomogeneous velocity field at the lower boundary of the infinite horizontal fluid layer. The main causes of convection in the problem under consideration are linear heat sources and the field of shear stresses. The main attention is paid to the study of the properties of the flow velocity field and the dependence of the structure of this field on the magnitude of the vertical swirl.

INTRODUCTION

One of the difficulties in finding the exact solutions to the Oberbeck–Boussinesq system of differential equations [1–4] (partial differential equations) is its nonlinearity. In addition, the properties of the solution are dependent on the boundary conditions, the physical parameters of the fluid, and environmental characteristics [5–10].

For shear flows, a closed system of equations describing the motion of a viscous fluid becomes overdetermined (the number of equations exceeds the number of unknown functions). Generalized classes of exact solutions allow finding solutions of overdetermined systems. Examples of such classes are given in [5–13].

This paper discusses the effect of constant vertical swirling on the properties of the flow velocity field in a boundary-value problem describing the fluid flow in an infinite horizontal layer, which is induced by specifying shear stresses on the free surface.

BOUNDARY VALUE PROBLEM FORMULATION

A system of equations of thermal convection in the Boussinesq approximation is considered. For steady shear flows of a viscous incompressible fluid, this system takes the form

$$V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y} = -\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2}\right);\ V_x \frac{\partial V_y}{\partial x} + V_y \frac{\partial V_y}{\partial y} = -\frac{\partial P}{\partial y} + \nu\left(\frac{\partial^2 V_y}{\partial x^2} + \frac{\partial^2 V_y}{\partial y^2} + \frac{\partial^2 V_y}{\partial z^2}\right);$$

$$\frac{\partial P}{\partial z} = g\beta T;\ V_x \frac{\partial T}{\partial x} + V_y \frac{\partial T}{\partial y} = \chi\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right);\ \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 0. \quad (1)$$

Here, $V_x = V_x(x,y,z)$, $V_y = V_y(x,y,z)$, and $V_z = 0$ are the components of the velocity vector $\boldsymbol{V}$; $P(x,y,z)$ is a deviation from hydrostatic pressure, divided by the average density; $T(x,y,z)$ is the deviation of temperature from the

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reference value; ν and χ are the kinematic (molecular) viscosity of the fluid and the coefficient of thermal diffusivity; β is the temperature coefficient of volumetric expansion of the fluid; g is acceleration of gravity.

The solution is sought in the class [7, 8]

$$V_x = U(z) + u(z)y;\ V_y = V(z);\ V_z = 0; \tag{2}$$

$$P = P_0(z) + P_1(z)x + P_2(z)y;\ T = T_0(z) + T_1(z)x + T_2(z)y. \tag{3}$$

Note that, when class (2) is substituted into the incompressibility equation (the last equation of system (1)), it is satisfied identically. Thus, the number of independent equations in this system and the number of unknown functions become equal.

In addition, for class (2) in the general case, all the components of the vorticity **Ω** = rot ***V*** are nonzero,

$$\mathbf{\Omega} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ V_x & V_y & 0 \end{vmatrix} = -\boldsymbol{i}\frac{\partial V_y}{\partial z} + \boldsymbol{j}\frac{\partial V_x}{\partial z} + \boldsymbol{k}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right) = \left(-\frac{\partial V}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial U}{\partial z} + \frac{\partial u}{\partial z}y\right)\boldsymbol{j} + (-u)\boldsymbol{k}.$$

System (1) taking into account representations (2) – (3) can be reduced to the following system of ordinary differential equations:

$$u'' = 0;\ T_1'' = 0;\ P_1' = g\beta T_1;\ \chi T_2'' = uT_1;\ P_2' = g\beta T_2;$$
$$\nu V'' = P_2;\ \nu U'' = Vu + P_1;\ \chi T_0'' = UT_1 + VT_2;\ P_0' = g\beta T_0. \tag{4}$$

Here, each of the functions included in system (4) depends only on the vertical variable z, the differentiation with respect to which is indicated by a prime.

We further investigate the case of constant vertical twist ($u = \Omega = const$). Let us consider the fluid flow in an infinite horizontal layer of thickness h in a gravity field, assuming that heat sources are given at both boundaries ($z = 0$ and $z = h$), and that the temperature at the lower boundary $z = 0$ is taken as the reference level,

$$T_0(0) = 0;\ T_1(0) = A;\ T_2(0) = 0;\ T_0(h) = \vartheta;\ T_1(h) = C;\ T_2(h) = 0. \tag{5}$$

We also assume that the lower boundary velocity is specified and that the uniform pressure (taken as the reference value) and the shear stress field are set on the upper one,

$$U(0) = V(0) = 0;\ u(0) = \Omega;\ P_0(h) = P_1(h) = P_2(h) = 0;\ \eta U'(h) = \xi_1;\ \eta V'(h) = \xi_2. \tag{6}$$

Here, η is the dynamic viscosity coefficient.

EQUATION SYSTEM SOLUTION

The solution of the boundary value problem (4) – (6) for the velocity field takes the form

$$u = \Omega;$$

$$V = \frac{\xi_2 h}{\eta}Z - \frac{g\beta\Omega h^5}{720\nu\chi}\left[A\left(14 - 15Z + 10Z^3 - 6Z^4 + Z^5\right) + C\left(16 - 15Z + 5Z^3 - Z^5\right)\right];$$

$$U = \frac{g\beta h^3}{720\nu}Z\left[A\left(4 - 6Z + 4Z^2 - Z^3\right) + C\left(8 - 6Z + Z^3\right)\right] +$$

$$+\frac{g\beta\Omega^2 h^7}{120960\nu^2\chi} Z\Big[A\left(528-392Z^2+210Z^3-56Z^5+24Z^6-3Z^7\right)+$$

$$+C\left(648-448Z^2+210Z^3-28Z^5+3Z^7\right)\Big]-\frac{\Omega\xi_2 h^3}{6\eta\nu} Z\left(3-Z^2\right)+\frac{\xi_1 h}{\eta} Z. \qquad (7)$$

Here, $Z = z / h$ is the dimensionless vertical coordinate.

INVESTIGATION OF THE SOLUTION

When a homogeneous heat source ($A = C = 0$) is specified in the case $\Omega = 0$, the velocities U and V are determined by the linear functions

$$U = \frac{\xi_1 h}{\eta} Z; \; V = \frac{\xi_2 h}{\eta} Z.$$

Thus, in the direction of both longitudinal axes, the flow reduces to a combination of unidirectional Couette-type flows [14].

In the general case ($A \neq 0$, $C \neq 0$, $\Omega \neq 0$), the velocity V in (7) can be represented in the following form:

$$V = \frac{Cg\beta\Omega h^5}{720\nu\chi} Z\Big[Z^5-5Z^3+15Z+a_1-a\left(Z^5-6Z^4+10Z^3-15Z+14\right)\Big], \qquad (8)$$

where $a = A/C$, $a_1 = -16+720\nu\chi\xi_2/(Cg\beta\eta\Omega h^4)$ are dimensionless parameters. The analysis of the zero points of polynomial (8) has shown that the velocity V can have two stagnant points (Fig. 1).

A similar study conducted for the velocity U allows us to conclude that the velocity U in (7) can have no more than four stagnant points inside the fluid layer under consideration (Fig. 2).

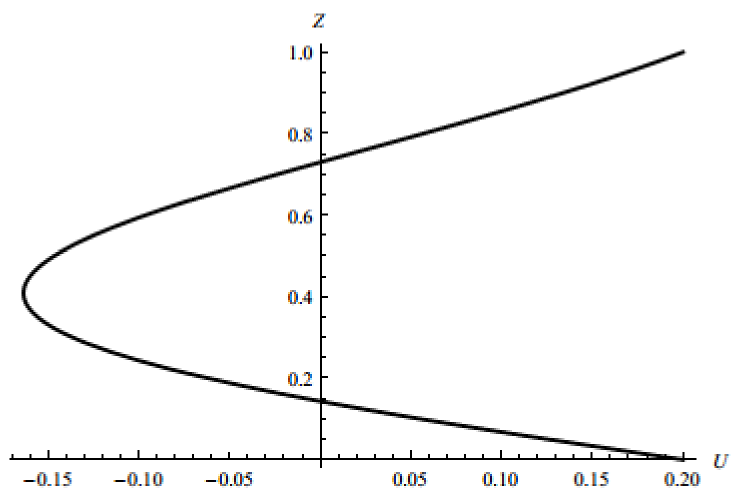

FIGURE 1. The profile of the velocity V for $a_1 = -15.2$, $a = -1.1$

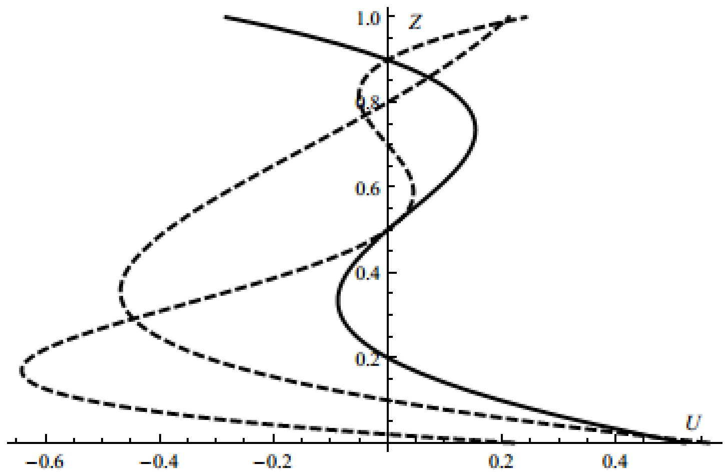

FIGURE 2. The profiles of the velocity U with different numbers of stagnant points

CONCLUSION

A new solution, describing the shear convection of a vertically swirling viscous incompressible fluid in an infinite horizontal layer, has been proposed. The fluid motion is induced by setting the heat sources and the shear stress field at the layer boundaries. It has been shown that no more than two stagnant points can exist in a fluid, although one of the velocity vector components can vanish up to four times in the layer bulk.

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